Relation between polymer and Fock excitations

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Abstract

To bridge the gap between background independent, non-perturbative quantum gravity and low energy physics described by perturbative field theory in Minkowski space-time, Minkowskian Fock states are located, analyzed and used in the background independent framework. This approach to the analysis of semi-classical issues is motivated by recent results of Varadarajan. As in that work, we use the simpler $U(1)$ example to illustrate our constructions but, in contrast to that work, formulate the theory in such a way that it can be extended to full general relativity.

Motivation

A detailed theory of quantum geometry was systematically developed in the mid-nineties [1-17]. Over the last two years, it was successfully applied to address some of the long-standing challenges of quantum gravity. These include: a statistical mechanical derivation of black hole entropy [18]; a resolution of the big-bang singularity in quantum cosmology [19]; and, a "finiteness result" in the path integral (or spin-foam) approach to quantum general relativity [20]. These results make a crucial use of the fundamental discreteness, predicted by the background independent quantum geometry. But this very discreteness has made it difficult to relate the underlying 'polymer excitations' of quantum geometry to the Fock states normally used in low energy physics. Candidate semi-classical states of quantum geometry *have been* introduced by a number of authors (for early ideas, see [21–24] and, for a detailed framework, [25]). However, there is no obvious relation between them and the Fock coherent states, normally used to analyze semi-classical issues in low energy physics. As a result, detailed answers to two key questions are yet to emerge: Can the background-independent, non-perturbative theory reproduce the familiar low-energy physics on, say, suitable coarse graining? and, Can one pin-point where and why perturbation theory fails?

More precisely the situation is as follows. Since the approach begins with general relativity coupled to matter (or, supergravity) and carries out a (canonical or path integral) quantization following standard procedures, at the simplest level it is clear that the quantum theory would have the correct classical limit. Similarly, the fact that the fundamental discreteness arises at the Planck scale [21] strongly suggests that while a coarse graining at, say, a TeV scale should reproduce the continuum physics, the ultraviolet behavior of the non-perturbative and perturbative theories would be very different. However, these can only be regarded as general indications. What is lacking is a systematic investigation leading to detailed answers to the two question.

Efforts at facing this challenge squarely have begun recently. A point of departure is provided by the recent work of Madhavan Varadarajan [26]. The purpose of this letter is outline the general approach and summarize some of the key results. Detailed proofs and further results will appear elsewhere [27].

The tension

In fully non-perturbative approaches to quantum gravity, semi-classical issues are conceptually difficult because there is no background space-time to begin with. Furthermore, the basic mathematical structures in these approaches are very different from those used in more familiar perturbative treatments. For example, in the quantum geometry framework, fundamental excitations are one dimensional, polymer-like [1-5]; a convenient basis of states is provided by spin-networks [7–10]; and, eigenvalues of the basic geometric operators defining triads [13], areas and volumes [12–14] are discrete. By contrast, in the Fock framework, the fundamental excitations are 3-dimensional, wavy; the convenient basis is labelled by the number n of gravitons, their momenta and helicities; and, all geometric operators have continuous spectra. The challenge is to bridge the gap between these apparently disparate frameworks. In most of this letter, for concreteness and simplicity, we will consider an $U(1)$ model which captures the key features of the problem at hand. However, our resolution of the tension between the two frameworks is well suited for generalization to full quantum gravity.

Let us begin by using the $U(1)$ example to bring out the tension in mathematical terms. Fix a space-like 3-plane M in Minkowski space. To construct the $U(1)$-analog of quantum geometry [1-5], let us begin by introducing the quantum configuration space $\bar{\mathcal{A}}$ for the $U(1)$ theory. An element $\bar{A}$ of $\bar{\mathcal{A}}$, called a generalized connection, associates with every oriented, analytical edge e in M a holonomy, i.e., an element of $U(1)$, such that: i) $\bar{A}(e^{-1}) = [\bar{A}(e)]^{-1}$; and, ii) if e_1, e_2 and $e_3 := e_1 \cdot e_2$ are all analytic edges, then $\bar{A}(e_3) = \bar{A}(e_1)\,\bar{A}(e_2)$.[1] The space $\bar{\mathcal{A}}$ carries a natural, diffeomorphism invariant measure μ_o, induced by the Haar measure on $U(1)$. Following the quantum geometry strategy [2–5], let us choose $\mathcal{H}_o := L^2(\bar{\mathcal{A}}, \mu_o)$ as the Hilbert space of states. There are two families of operators defined naturally on $\mathcal{H}_o$. The first, $\hat{h}_\gamma$, are *holonomy operators*, labelled by closed loops γ. They are unitary and operate by multiplication. The second family, $\hat{E}_S$, is labelled by 2-surfaces S without boundary. These are self-adjoint operators, representing the flux of the electric field through S. Thus, in this framework, the connection is smeared along 1-dimensional loops and the electric field along 2-surfaces.

Eigenvalues of the electric flux operators are integers; *in this representation, the electric flux is quantized.* The eigenstates, called *flux networks*, provide a useful orthonormal basis

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[1]It is often convenient to quotient $\bar{\mathcal{A}}$ by the action of generalized gauge group $\bar{\mathcal{G}}$ consisting of (arbitrarily discontinuous) maps from M to $U(1)$. The resulting space $\bar{\mathcal{A}}/\bar{\mathcal{G}}$ is also the Gel'fand spectrum of the obvious holonomy algebra constructed from smooth connections on M [1].

in $\mathcal{H}_o$. Given any graph α with N oriented edges, and an assignment of an integer to each edge, we can define a flux network function $\mathcal{N}_{\alpha,\vec{n}}$ on $\bar{\mathcal{A}}$ via

$$\mathcal{N}_{\alpha,\vec{n}}\,(\bar{A}) := [\bar{A}(e_1)]^{n_1}\ldots[\bar{A}(e_N)]^{n_N}. \tag{1}$$

Given any 2-surface S, the action of $\hat{E}_S$ on this state is given by [13]:

$$\hat{E}_S\,\mathcal{N}_{\alpha,\vec{n}} = \left(\frac{\hbar}{2}\sum_I k_I\,\epsilon_I\,n_I\right)\mathcal{N}_{\alpha,\vec{n}} \tag{2}$$

where the sum is over edges of α which intersect S; $k_I = \pm 1$ depending on whether the Ith edge lies above or below S and $\epsilon_I = \pm 1$ depending on whether the Ith edge is oriented to leave S or arrive at S. Thus, the labelled edges of α can be regarded as flux lines of the electric field operator, the integer n_I denoting the number of 'fluxons' along the Ith edge. Being 1-dimensional , these excitations are said to be 'polymer-like'.

We will refer to finite linear combinations of flux networks as cylinder functions and denote by Cyl the dense subspace of $\mathcal{H}_o$ they span.[2] The algebraic dual of Cyl will be denoted by Cyl*. We thus have a triplet,

$$\mathrm{Cyl} \subset \mathcal{H}_o \subset \mathrm{Cyl}^\star\,, \tag{3}$$

which is analogous to the Gel'fand triplet, often used in quantum mechanics [28]. Quantum geometry has analogous structures: There, spin networks are eigenstates of geometric operators, the kinematical Hilbert space $\mathcal{H}_o$ enables one to obtain well-defined operators representing quantum constraints, and physical states of quantum gravity –i.e., solutions to the constraints– belong to Cyl*. Note that, as in quantum geometry, the entire framework –states as well as the operators– is diffeomorphism invariant; in particular, the space-time metric is not used anywhere.

The Fock framework, on the other hand, makes a heavy use of the space-time metric. For our purposes, it is more convenient to use the Schrödinger representation. The Hilbert space $\mathcal{H}_F$ is then given by $\mathcal{H}_F = L^2(\mathcal{S}^\star, d\mu_F)$, where $\mathcal{S}^\star$ is the space of tempered distributions on M and μ_F is the Gaussian measure (defining the usual Fock space of photons). The basic operators are now the smeared connection and electric fields, $\hat{A}(f)$ and $\hat{E}(g)$. The connection and the electric field are both transverse operator-valued distributions and require a *3-dimensional* smearing. Their spectra are continuous.

The challenge is to relate the two descriptions which appear to have little in common. The natural 'homes' for these quantum theories are the quantum configuration spaces, $\bar{\mathcal{A}}$ and $\mathcal{S}^\star$. They are quite different spaces; while they admit a non-trivial intersection, neither space is contained in the other. Therefore, at first sight it seems difficult to even begin a comparison.

[2]This is a slightly different usage of the term from [2–6] but more convenient for our purposes here.

A Resolution

A primary source of the tension between the two frameworks lies in the fact that while holonomies play a central role in the definition of generalized connections in $\bar{\mathcal{A}}$, the holonomy of a general element of $\mathcal{S}^\star$ fails to be well defined because tempered distributions have to be smeared in *three* dimensions, while the holonomy provides smearing along only one dimension [29]. To overcome this incompatibility, let us define a 'contraction map' $\mathcal{C}_r$ on $\mathcal{S}^\star$. The Fourier transform of a tempered distribution is again a tempered distribution and it is easier to define the action of $\mathcal{C}_r$ in the momentum space:

$$\mathcal{C}_r : \mathcal{S}^\star \mapsto \mathcal{S}^\star; \qquad \mathcal{C}_r(A_a(\vec{k})) = e^{-\frac{1}{2}k^2r^2}\, A_a(\vec{k}) \;=: A_a^{(r)}(\vec{k}) \tag{4}$$

where r is any fixed positive real number. Let us denote by $A_a^{(r)}(x)$ the Fourier transform of $A_a^{(r)}(k)$. A simple calculation shows that, given any smooth loop γ, the holonomy of the connection $A_a^{(r)}(x)$ around γ is well-defined. Thus, $\mathcal{C}_r$ *tames* tempered distribution sufficiently to make their line-integrals well-defined. Therefore, intuitively, one would expect that every $A_a^{(r)}(x)$ defines an element of $\bar{\mathcal{A}}$. Using the characterization of $\bar{\mathcal{A}}$ in terms of the so-called 'hoop group' [2], one can show that this expectation is correct. Thus, $\mathcal{C}_r$ is a 1-1, onto mapping from $\mathcal{S}^\star$ to $\mathcal{S}^\star \cap \bar{\mathcal{A}}$.

We can now push forward the Fock-measure μ_F on $\mathcal{S}^\star$ to obtain a measure $\mu_F^{(r)}$ on $\bar{\mathcal{A}}$ (which is faithful on $\mathcal{S}^\star \cap \bar{\mathcal{A}}$ but not on $\bar{\mathcal{A}}$). Consequently, Fock states can now be represented as functions on $\bar{\mathcal{A}}$ which are square-integrable with respect to $\mu_F^{(r)}$. Of course, the actions of the Fock operators $\hat{A}(f)$ and $\hat{E}(f)$ on this representation are more involved [26], obtained from the standard actions via pull-backs and push-forwards defined by $\mathcal{C}_r$. This complication is inevitable; if one wishes to retain the standard action, it is impossible to have well-defined holonomy operators in the Fock representation [29]. Nonetheless, for *each* $r > 0$, $L^2(\bar{\mathcal{A}}, d\mu_F^{(r)})$ provides a representation of the usual operator algebra which is unitarily equivalent to the standard Fock representation.

The conceptual simplification brought about by this step is that $\bar{\mathcal{A}}$ *can serve as a 'common home' for both the 'polymer' and the Fock descriptions*. We can now meaningfully ask for the relation between them.

The relation between measures

Since both μ_o and $\mu_F^{(r)}$ are measures on the same space $\bar{\mathcal{A}}$, we can now ask for the relation between them, and, more generally, investigate the relation between Fock states and elements of the triplet $\mathrm{Cyl} \subset \mathcal{H}_o \subset \mathrm{Cyl}^\star$ in the polymer description.

Given a graph α, there is an associated quantum configuration space $\mathcal{A}_\alpha$, isomorphic to $[U(1)]^N$, where N is the number of edges of α. $\mathcal{A}_\alpha$ is obtained simply by restricting the action of generalized connections in $\bar{\mathcal{A}}$ to the edges of the graph α and can be regarded as the configuration space of an $U(1)$ lattice gauge theory based on α. The Haar measure on $U(1)$ provides a natural measure μ_α^o on $\mathcal{A}_\alpha$ and the flux network states $\mathcal{N}_{\alpha,\vec{n}}$ provide an orthonormal basis for the Hilbert space $H_\alpha^o := L^2(\mathcal{A}_\alpha, d\mu_\alpha^o)$. $\bar{\mathcal{A}}, \mu_o$ and $\mathcal{H}_o$ can all be obtained as projective limits of $\mathcal{A}_\alpha, \mu_\alpha^o, H_\alpha^o$ in which one considers larger and larger graphs α [3–6].

Now, *every* measure on $\bar{\mathcal{A}}$ arises as a consistent family of measures on $\mathcal{A}_\alpha$ associated with graphs α [3–6]. This is in particular true of the measure $\mu_F^{(r)}$. Therefore, to understand the relation between Fock and polymer excitations, we can compare the measures μ^o_α and $\mu^{(r)}_\alpha$ on $\mathcal{A}_\alpha$, corresponding to the measures μ_o and $\mu^{(r)}_F$ on $\bar{\mathcal{A}}$. A simple calculation shows that, for any graph α, the measure $\mu^{(r)}_\alpha$ is absolutely continuous with respect to μ^o_α.

To exhibit the function relating them, we first recall [30] the notion of the r-form factor associated with any oriented edge e. The form-factor $F^a_e(\vec{x})$ of e is given by

$$F^a_e(\vec{x}) := \int_e ds\, \dot{e}^a(s)\ \delta^3(\vec{x}, \vec{e}(s)) \tag{5}$$

so that the holonomy of any connection $A^{(r)}$ along e is given by $\int d^3x A^{(r)}_a(\vec{x}) F^a_e(\vec{x})$. (Note that $F^a_e(\vec{x})$ is insensitive to orientation-preserving re-parameterizations $s \rightarrow s'$ of e.) The 'tamed' r-form factor is defined as:

$$F^a_{e,r}(\vec{x}) := \frac{1}{(2\pi)^{\frac{3}{2}}}\ \int_e ds\, \dot{e}^a(s)\ e^{-\frac{|\vec{x}-\vec{e}(s)|^2}{2r^2}}$$

Denote by $F^a_e(\vec{k})$ the Fourier transform of $F^a_e(\vec{x})$. Then the Fourier transform of the 'tamed' r-form factor is simply:

$$F^a_{e,r}(\vec{k}) := e^{-\frac{1}{2}k^2r^2}\ F^a_e(\vec{k})$$

In terms of these r-Form factors, the two measures are related by

$$d\mu^{(r)}_\alpha = \left[\sum_{\vec{n}} e^{-\frac{\hbar}{4}\sum_{I,J} G_{IJ}\, n_I n_J}\ \mathcal{N}_{\alpha,\vec{n}}(\bar{A}_\alpha)\right]\ d\mu^o_\alpha \tag{6}$$

where I, J range from 1 to N (the number of edges of α), and

$$G_{IJ} = \int \frac{d^3k}{|k|}\ \bar{F}_{e_I,r}\ \cdot\ F_{e_J,r}\ , \tag{7}$$

the $\cdot$ denoting the contraction of the vector indices of form-factors with the natural Euclidean metric on M. The measure μ^o_α knows nothing about space-time geometry while the measure $\mu^{(r)}_\alpha$ does. This information (as well as our 'taming procedure') is neatly coded in the $N\times N$ matrix G_{IJ} on the space of edges of α. One can show that the infinite sum in the square brackets on the right side of (6) converges to a continuous function φ_α on (the compact space) $\mathcal{A}_\alpha$. Thus, given any graph, the two measures are absolutely continuous with respect to one another. However, the family of functions φ_α obtained by varying the graph α is not consistent: if $\alpha > \tilde{\alpha}$, φ_α does not equal the pull-back of $\varphi_{\tilde{\alpha}}$ under the natural projection map from $\mathcal{A}_\alpha$ to $\mathcal{A}_{\tilde{\alpha}}$. Therefore, the measures μ_o and $\mu^{(r)}_F$ on $\bar{\mathcal{A}}$ fail to be absolutely continuous; they are inequivalent. Physically this inequivalence is to be expected. The above discussion serves to pinpoint how the difference arises mathematically.

Finally, it is interesting to note that the Fock measure $\mu^{(r)}_F$ can be obtained by taking a projective limit of measures $\mu^{(r)}_\alpha$ associated with (floating) lattices α. This is an interesting and, for our purposes, crucial alternative to the conventional procedure in which one begins with a fixed lattice and lets the lattice spacing go to zero.

Fock states as elements of Cyl*

Each regular measure on $\bar{\mathcal{A}}$ naturally gives rise to a representation of the holonomy $C^\star$ algebra [1,2]. Since the measures μ_o and $\mu_F^{(r)}$ are inequivalent, so are the two representations. Therefore, Fock states can not be realized as elements of $\mathcal{H}_o$. In particular, there is no state in $\mathcal{H}_o$ in which the expectation values of all holonomy operators equal those in the vacuum state in $\mathcal{H}_F^{(r)}$. Nonetheless, as shown below, every coherent state (and hence every state) in $\mathcal{H}_F^{(r)}$ can be naturally regarded as an element of Cyl*. In full quantum gravity, solutions to constraints naturally lie in Cyl*. The fact that Fock states used in the low energy, perturbative analyses also share this 'home' will facilitate the comparison between non-perturbative and perturbative treatments.

In [26], Varadarajan imposed the 'poincaré invariance condition' to single out, among elements of Cyl*, the vacuum state of $\mathcal{H}_F^{(r)}$. His calculation was tailored to the framework developed in [30] and thus used closed loops. For our purposes, it is more convenient to use flux network states. Note first that since every element of Cyl can be expressed as a finite linear combination of flux network states, there is a natural basis in Cyl* consisting of elements of the type $<\mathcal{N}_{\alpha,\vec{n}}|$ which maps the flux network function $|\mathcal{N}_{\alpha,\vec{n}}>$ to one and every other flux network function to zero [11]. In terms of this basis, the element $<V_F^{(r)}|$ of Cyl* representing the vacuum in $\mathcal{H}_F^{(r)}$ can be written as follows:

$$<V_F^{(r)}| = \sum_{\alpha,\vec{n}} \left[e^{-\frac{\hbar}{2}\sum_{IJ} G_{IJ} n_I n_J}\right] <\mathcal{N}_{\alpha,\vec{n}}| \ . \tag{8}$$

Although the sum is over an uncountable set, while acting on any element of Cyl, only a finite number of terms are non-zero, whence the action is well-defined.[3]

Note some interesting features of this construction.

i) Cyl* and the basis $<\mathcal{N}_{\alpha,\vec{n}}|$ in it are constructed in a diffeomorphism invariant fashion; these structures know nothing about Minkowski geometry. How is it then that we can locate Minkowskian Fock states as elements of Cyl*? The information about space-time geometry gets fed in to the expression (8) through the coefficients in the linear combination of these basis vectors.

ii) These numerical coefficients have a non-local spatial dependence; there is an 'interaction' between all different edges of the graph α with each other. These non-local correlations are characteristic of Fock states and have physical consequences. In particular, this non-locality is responsible for the fact that the Fock vacuum represents the 'Coulomb phase' of the $U(1)$ theory in which Wilson and (their dual) 't Hooft loops both go as exponentials of the length of the loop rather than area. In the polymer representation, one *can* define semi-classical states without such non-local correlations [25]. But such states typically belong to the 'Higgs phase' ('dual' of the confined phase) in which the Wilson loop goes as the exponential of

[3] The exponents on the right sides of (6) and (8) are closely related but differ by a factor of 2: while the expression in the square brackets in (6) is the vacuum expectation value of the operator $\hat{\mathcal{N}}_{\alpha,\vec{n}}$ (which acts by multiplication), that in (8) is the action of the element $<V_F^{(r)}|$ of Cyl* on the cylindrical function $\mathcal{N}_{\alpha,\vec{n}}$.

the length but the 't Hooft loop goes as the exponential of the area.
iii) Since $\mathrm{Cyl}^\star$ is the algebraic dual on Cyl, the holonomy and electric flux operators $\hat{h}_\gamma$ and $\hat{E}_S$ on Cyl have a natural, well-defined action on $\mathrm{Cyl}^\star$ by duality. One can show that the subspace of $\mathrm{Cyl}^\star$ obtained by acting repeatedly by $\hat{h}_\gamma$ on $< V_F^{(r)} |$ is the embedding of a dense subspace of $\mathcal{H}_F^{(r)}$ in $\mathrm{Cyl}^\star$. In this sense, all Minkowskian, photon Fock states are realized as elements of $\mathrm{Cyl}^\star$. However, $\mathrm{Cyl}^\star$ is *very large*: not only does it contain such perturbative states on other space-time geometries but it also contains states (such as the basis used in (8)) which lie *entirely outside the semi-classical regime*. In the context of full quantum gravity, such states in $\mathrm{Cyl}^\star$ would represent quantum excitations of the geometry and the $U(1)$ gauge field which can not be described in space-time terms.
iv) It is because the *photon Fock states span a 'very small' sub-space of* $\mathrm{Cyl}^\star$ that the Fock representation fails to capture flux quantization —i.e., fundamental discreteness— of the 'polymer' representation. For, while the basis $< \mathcal{N}_{\alpha,\vec{n}} |$ in $\mathrm{Cyl}^\star$ provides infinitely many eigenvectors with discrete eigenvalues of the electric flux operators $\hat{E}_S$, none of them belong to the image of $\mathcal{H}_F^{(r)}$ in $\mathrm{Cyl}^\star$. Indeed, although the operators $\hat{E}_S$ have a well-defined action on $\mathrm{Cyl}^\star$, *they fail to leave the image of* $\mathcal{H}_F^{(r)}$ *in* $\mathrm{Cyl}^\star$ *invariant.* From the 'polymer' perspective this is why the flux operators fail to be well-defined in the Fock space.

Finally, we can represent any coherent state as an element of $\mathrm{Cyl}^\star$. Using the fact that coherent states are eigenstates of annihilation operators, one can extend the above procedure, used for the vacuum state, to represent a general coherent state, peaked at a smooth transverse connection A_o and electric field E_o, as an element $< \Psi^{(r)}_{(A_o,E_o)} |$ of $\mathrm{Cyl}^\star$:

$$< \Psi^{(r)}_{(A_o,E_o)} | = \sum_{\alpha,\vec{n}} \left[e^{i\hbar \sum_I n_I \int d^3k\, \bar{F}_{I,r}(k)\cdot f(k)}\; e^{-\frac{\hbar}{2}\sum_{IJ} G_{IJ}\, n_I n_J} \right] < \mathcal{N}_{\alpha,\vec{n}} | \, , \tag{9}$$

where, $\hbar f(k) = A_o(k) - (i/\mid k \mid) E_o(k)$. Again the information about all background fields —the Minkowskian geometry as well as the electromagnetic field (A_o, E_o) which approximates the semi-classical state— is coded in the coefficients of the linear combination. As a result, although the space $\mathrm{Cyl}^\star$ and the basis $< \mathcal{N}_{\alpha,\vec{n}} |$ is constructed in a non-perturbative, diffeomorphism invariant fashion, individual elements of $\mathrm{Cyl}^\star$ can still carry information about background fields, needed in Minkowskian, low energy perturbation theory.

Shadow states

At this stage of development of quantum geometry, $\mathrm{Cyl}^\star$ does not have a natural inner product with respect to which both the eigenstates $< \mathcal{N}_{\alpha,\vec{n}} |$ of the electric flux operators and the (images of the) Fock states are normalizable. One obvious strategy is to simply ignore those elements of $\mathrm{Cyl}^\star$ which do not represent Fock states and use the standard Fock norm (provided by $\mu_F^{(r)}$) on those which do. But then the non-perturbative, 'polymer' perspective would be entirely lost and one would just be reproducing the Fock representation in an unnecessarily complicated fashion. What we need is a new structure which would enable us to analyze the physical content of Fock coherent states *from the non-perturbative perspective.* This will provided by the notion of shadow states.

For concreteness, let us focus on the vacuum $< V_F^{(r)} |$. It follows from the expression (8) that, given *any* flux network function $\mathcal{N}_{\beta,\vec{n}}$ associated with a fixed graph β, the action of $< V_F^{(r)} |$ on $\mathcal{N}_{\beta,\vec{n}}$ can be written as:

$$<V_F^{(r)} \mid \mathcal{N}_{\beta,\vec{n}}> = e^{-\frac{\hbar}{2}\sum_{IJ} G_{IJ} n_I n_J} := \int_{\mathcal{A}_\beta} d\mu_\beta^o \, \bar{V}_\beta^{(r)} \, \mathcal{N}_{\beta,\vec{n}} \tag{10}$$

where I, J now label the edges of the graph β and where the function $V_\beta^{(r)}(\bar{A}_\beta)$ on $\mathcal{A}_\beta$ is given by

$$V_\beta^{(r)}(\bar{A}_\beta) = \sum_{\vec{n}} e^{-\frac{\hbar}{2}\sum_{IJ} G_{IJ} n_I n_J} \, \mathcal{N}_{\beta,\vec{n}}(\bar{A}_\beta) \tag{11}$$

This implies that the action of the element $<V_F^{(r)} \mid$ of Cyl* on *any* element φ_β of Cyl (based on β) is the same as the inner product of $V_\beta^{(r)}(\bar{A})$ with φ_β (on $L^2(\mathcal{A}_\beta, d\mu_\beta^{(r)})$). Therefore, we will refer to the function $V_\beta^{(r)}$ on $\mathcal{A}_\beta$ as the *shadow of the Fock vacuum* $<V_F^{(r)} \mid$ on the graph β. The set of shadows on all possible graphs captures the full information in $<V_F^{(r)} \mid$. Furthermore, since each shadow is an element of a *Hilbert space* $L^2(\mathcal{A}_\beta, d\mu_\beta^{(r)})$ associated with *some* graph β, we can now take expectation values and and calculate fluctuations of operators in the shadow states. The construction can be repeated for any Fock coherent state. Therefore, using shadow states we can spell out the precise sense in which Fock coherent states are semi-classical also from the non-perturbative perspective and to understand limitations of the Fock description in the ultraviolet regime [27].

Heat kernels and quantum gravity

Let us now return to non-perturbative gravity and indicate how our $U(1)$ constructions can be extended to that case. Our goal is to find an element $< \mathcal{M}^{(r)} \mid$ of gravitational Cyl* which represents a semi-classical state peaked at the trivial Minkowskian initial data (constant triads ${}^0E_i^a$ and zero gravitational connection, $A_a^i = 0$). Now the gauge group is $SU(2)$ and flux network states are replaced by the spin network states $\mathcal{N}_{\alpha,\vec{j},\vec{I}}(\bar{A})$, where half-integers j_e (or, irreducible unitary representations of $SU(2)$) label edges e of the graph, and invariant operators I_v on (the tensor product of 'incoming and outgoing' representations of) $SU(2)$ label its vertices v [8–11]. In spite of the fact that the situation is now technically more complicated, every element of Cyl* can be still expanded in terms of the dual spin network basis, $<\mathcal{N}_{\alpha,\vec{j},\vec{I}}\mid$. Our goal is to find the expansion coefficients of $< \mathcal{M}^{(r)} \mid$ in this basis.

To carry out this task let us first reformulate the construction of the shadows of the U(1) Fock vacuum using 'heat kernel' ideas [6,31] which can be then extended directly to the non-Abelian context. Let us begin by introducing a family of operators. Fix a vertex v of a graph β and label by I the edges which have v as an endpoint. Define operators X_I via their action on any cylindrical function φ_β associated with β:

$$(X_I \, \varphi_\beta) = \begin{cases} -(\bar{A}(e_I) \, \partial\varphi_\beta / \partial(\bar{A}(e_I)) \,, & \text{if } v \text{ is the target of } e_I, \\ (\bar{A}(e_I) \, \partial\varphi_\beta / \partial(\bar{A}(e_I)) \,, & \text{if } v \text{ is the source of } e_I, \end{cases} \tag{12}$$

where on the right side we have regarded φ_β as a function of the N holonomies $\bar{A}(e_1), \ldots, \bar{A}(e_N)$. Thus, if v is the target, X_I is the right invariant vector field on the copy of $U(1)$ associated with the Ith edge and, if it is the source, the left invariant vector field. Using the fact that the flux networks are eigenstates of X_I, the shadow vacuum $V_\beta^{(r)}$ of (11) can be expressed as:

$$V^{(r)}_\beta(\bar A_\beta) = \sum_{\vec n} e^{\frac{\hbar}{8}\sum_{v,v'}\sum_{I,I'} G_{II'} X_I X_{I'}}\ \mathcal{N}_{\beta,\vec n}(\bar A_\beta) \tag{13}$$

where, I and I' now label edges associated with vertices v and v' respectively. The exponent is a negative definite self-adjoint operator on $L^2(\mathcal{A}_\beta, d\mu^o_\beta)$, reminiscent of heat kernels.[4] Indeed, by using the Peter-Weyl expansion of the Dirac δ distribution on $\mathcal{A}_\beta \simeq [U(1)]^N$, one can re-express more conveniently as

$$V^{(r)}_\beta(\bar A_\beta) = e^{\frac{\hbar}{8}\sum_{v,v'}\sum_{I,I'} G_{II'} X_I X_{I'}}\ \delta_0(\bar A_\beta) \tag{14}$$

where $\delta_0(\bar A_\beta)$ is the Dirac-distribution on $(\mathcal{A}_\beta, d\mu^o_\beta)$ peaked at the zero connection. On any fixed graph β, the right side is a continuous function on $\mathcal{A}_\beta$. However, as remarked earlier, the family of functions obtained by varying graphs fails to be consistent. On the other hand, one can show that the family of operators on the right side of (14) *is* consistent whence the right side provides a consistent family of *distributions*. Hence, the element of Cyl* representing the Fock vacuum can be written as

$$< V^{(r)}_F \mid \varphi > = \int_{\bar{\mathcal{A}}} d\mu_o\ (e^\Theta\ \delta_o(\bar A))\ \varphi \qquad\qquad \forall \varphi \in \mathrm{Cyl} \tag{15}$$

where Θ is the projctive limit of the family of operators $[\frac{\hbar}{8}\sum_{v,v'}\sum_{I,I'} G_{II'} X_I X_{I'}]$ on $(\mathcal{A}_\beta, d\mu^o_\beta)$ and $\delta_0(\bar A)$ is the Dirac-distribution on $(\bar{\mathcal{A}}, d\mu_o)$ peaked at the zero generalized connection. Similarly, the coherent state (9) can be expressed as

$$< \Psi^{(r)}_{(A_o,E_o)} \mid\ \varphi > = \int_{\bar{\mathcal{A}}} d\mu_o\ (e^\Theta\ \delta_{\tilde f}(\bar A))\ \varphi \qquad\qquad \forall \varphi \in \mathrm{Cyl} \tag{16}$$

where $\delta_{\tilde f}$ is the Dirac-distribution peaked at $\tilde f = A_o(k) - (i/\mid k\mid) E_o(k)$, regarded as as a (complex) generalized connection.

We will now illustrate how these considerations can be extended to quantum gravity. To carry out this task, U(1) has to be replaced by SU(2). This is a non-trivial task because of the subtleties associated with non-Abelian gauge invariance. Specifically, the $U(1)$ operators X_I are now replaced by $SU(2)$ operators J^i_I where i is an su(2) Lie-algebra index [6,13] and, to construct the analog of the operator Θ, we must find a way to contract these su(2) indices associated with *different vertices*. This would have been a major obstacle without recourse to a background connection to transport the indices between different vertices. Fortunately, however, since our task is to construct semi-classical states peaked at given classical fields (A^i_a, E^a_i), we can use the given A^i_a as the background connection. Then, for $< \mathcal{M}^{(r)} \mid$, the required parallel transport is trivial since the background connection vanishes!

Since the state is to be peaked at a constant triad ${}^0E^a$, and since the triad and the connection have *different* physical dimensions, we need to introduce a new length scale ℓ

[4]If $G_{II'}$ were $\delta_{II'}$, the exponent would simply be the product of Laplacians, one for the copy of the group associated with each edge and $V^{(r)}_\beta$ would be a group coherent state on $[U(1)]^N$ a la Hall [32].

(to be fixed by the Planck length $\ell_{\rm Pl}$ and the macroscopic scale L of physical interest; see below). Our candidate shadow states $<\mathcal{M}^{(r)}_\beta\,|$ are thus given by

$$\mathcal{M}^{(r)}_\beta(\bar{A}_\beta) = e^{\frac{\hbar}{8}\sum_{v,v'}\sum_{I,I'} G_{II'}\, K_{i,i'}\, J^i_I J^{i'}_{I'}}\ \delta_{{}^0E}(\bar{A}_\beta)\,. \tag{17}$$

Here $\delta_{{}^0E}(\bar{A})$ is the Dirac-distribution peaked at $-(i/\ell)\,{}^0E^i_a$ (regarded as a connection on M in the gauge specified by the background fields under consideration), $K_{i,i'}$ is the Cartan-Killing metric on su(2) (and the internal indices are transported between vertices v and v' by the trivial connection). Again, as the graph varies, operators on the right hand side yield a consistent family. Hence, the right side of (17) constitutes a consistent family of distributions on the configuration spaces $(\mathcal{A}_\beta, d\mu^o_\beta)$ and thus defines an element $<\mathcal{M}^{(r)}\,|$ of Cyl*. Since our specification of the background field ($A^i_a = 0$) is not gauge invariant, neither is the state $<\mathcal{M}^{(r)}\,|$. However, if we wish, we can easily group average it over the group $\mathcal{G}$ of (arbitrarily discontinuous) gauge transformations and obtain a gauge invariant state in Cyl* [11].

Properties of this candidate semi-classical state and its generalizations for other background classical gravitational fields (A^i_a, E^a_i) are being investigated.

Statistical geometry

In practice —particularly for numerical semi-classical calculations which were recently launched— it is awkward to have to deal with *all* graphs. Furthermore it is clear that, to probe a semi-classical state effectively, the graph should be sufficiently fine; otherwise the shadow state would be too crude an approximation. For semi-classical purposes, then, can one restrict oneself to a judiciously chosen sub-family which is small enough to be manageable and yet large enough for the associated shadows to capture all the relevant information contained in the semi-classical state in Cyl* from which they originate? The answer is in the affirmative. We will summarize this strategy from the perspective of quantum gravity, coupled to Maxwell theory; for details, see [33,27].

Given a 3-manifold M with a positive-definite metric q^o_{ab}, using well established techniques from statistical geometry, one can introduce on it a natural family of (Voronoi) graphs [33]. For simplicity, let us suppose M is a 3-torus, q^o_{ab} is flat and endows M with a volume V. Consider a random sprinkling of points in M with a given mean density ρ. Then there is a natural procedure to construct a simplicial complex and a dual cell complex, and introduce, from the cell complex, a graph $\alpha_{x_1,\ldots,x_n}$, labelled by the $n = V\rho$ points of the given sprinkling (and of course (M, q^o_{ab}, ρ)). The construction is *covariant* in the sense that it does not require any additional inputs. In particular, then, the family of graphs $\{\alpha_{x_1,\ldots,x_n}\}$ is preserved by the action of isometries of q^o_{ab} on M. For large n, almost all vertices of graphs are four-valent, whence they are especially well-suited for quantum geometry [14]. Furthermore, using techniques from statistical geometry, one can estimate the number of vertices in any given 'sufficiently large' region and the number of intersections of the graph with any 'sufficiently large' surface with slowly varying extrinsic curvature. These estimates facilitate the task of calculating expectation values and fluctuations of geometric and Maxwell operators in candidate shadow states based on these Voronoi graphs. Thus, this family appears to be large enough to capture 'enough' shadow states and yet small enough to be manageable.

We will conclude by summarizing the qualitative indications that have been obtained from these and related calculations. First, one loop QED corrections to the Maxwell vacuum $< V_F^{(r)} |$ in Minkowski space have been calculated by Dreyer and Ghosh [34]. Their result shows that the parameter r is related to the cut-off, tending to zero as the ultra-violet cut off in the momentum space goes to infinity. On the other hand, quantum geometry (without Maxwell fields) has also been examined using Voronoi graphs [33]. One finds that no state based on this graph can serve as the shadow of the semi-classical state in $\mathrm{Cyl}^\star$ peaked at q^o_{ab}, unless the mean separation $a = 1/\sqrt[3]{\rho}$ between its vertices is greater than (a certain multiple of) the Planck length ℓ_{Pl} [30,33]. Thus, quantum geometry has a built-in cut off at ℓ_{Pl}, whence we are led to set $r \geq \ell_{\mathrm{Pl}}$. Suppose we are interested in observables associated with a macroscopic scale L. (For example, we may be interested in measuring areas or fluxes of electric and magnetic fields across surfaces of a characteristic length greater than or equal to L) Then, from the 'polymer perspective', one finds that the Fock description is adequate only if $L >> r \geq \ell_{\mathrm{Pl}}$. For questions involving frequencies comparable to or greater than the Planck frequency, the Fock description is a poor approximation to the 'fundamental', polymer description. It seems very likely that sharpened versions of these calculations will lead to detailed answers to the two questions with which we began.

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